A linear-time algorithm for the strong chromatic index of Halin graphs

Ton Kloks[1]* and Yue-Li Wang[2]

[1] Department of Computer Science
National Tsing Hua University, No. 101, Sec. 2, Kuang Fu Rd., Hsinchu, Taiwan
kloks@cs.nthu.edu.tw
[2] Department of Information Management
National Taiwan University of Science and Technology
No. 43, Sec. 4, Keelung Rd., Taipei, 106, Taiwan
ylwang@cs.ntust.edu.tw

Abstract. We show that there exists a linear-time algorithm that computes the strong chromatic index of Halin graphs.

1 Introduction

Definition 1. *Let* $G = (V, E)$ *be a graph. A* strong edge coloring *of* G *is a proper edge coloring such that no edge is adjacent to two edges of the same color.*

Equivalently, a strong edge coloring of G is a vertex coloring of $L(G)^2$, the square of the linegraph of G. The strong chromatic index of G is the minimal integer k such that G has a strong edge coloring with k colors. We denote the strong chromatic index of G by $s\chi'(G)$.

Recently it was shown that the strong chromatic index is bounded by

$$(2 - \epsilon)\Delta^2$$

for some $\epsilon > 0$, where Δ is the maximal degree of the graph [12].[3] Earlier, Andersen showed that the strong chromatic index of a cubic graph is at most ten [1].

Let $\mathcal{G}$ be the class of chordal graphs or the class of cocomparability graphs. If $G \in \mathcal{G}$ then also $L(G)^2 \in \mathcal{G}$ and it follows that the strong chromatic index can be computed in polynomial time for these classes. Also for graphs of bounded treewidth there exists a polynomial time algorithm that computes the strong chromatic index [13].[4]

* This author is supported by the National Science Council of Taiwan, under grant NSC 99–2218–E–007–016.

[3] In their paper Molloy and Reed state that $\epsilon \geqslant 0.002$ when Δ is sufficiently large.

[4] This algorithm checks in $O(n(s+1)^t)$ time whether a partial k-tree has a strong edge coloring that uses at most s colors. Here, the exponent $t = 2^{4(k+1)+1}$.

Definition 2. *Let* T *be a tree without vertices of degree two. Consider a plane embedding of* T *and connect the leaves of* T *by a cycle that crosses no edges of* T*. A graph that is constructed in this way is called a* Halin graph*.*

Halin graphs have treewidth at most three. Furthermore, if G is a Halin graph of bounded degree, then also $L(G)^2$ has bounded treewidth and thus the strong chromatic index of G can be computed in linear time. Recently, Ko-Wei Lih, *et al.*, proved that a cubic Halin graph other than one of the two 'necklaces' Ne_2 (the complement of C_6) and Ne_4, has strong chromatic index at most 7. The two exceptions have strong chromatic index 9 and 8, respectively. If T is the underlying tree of the Halin graph, and if $G \neq Ne_2$ and G is not a wheel W_n with $n \neq 0 \bmod 3$, then Ping-Ying Tsai, *et al.*, show that the strong chromatic index is bounded by $s\chi'(T)+3$. (See [14, 15] for earlier results that appeared in regular papers.[5])

If G is a Halin graph then $L(G)^2$ has bounded rankwidth. In [5] it is shown that there exists a polynomial algorithm that computes the chromatic number of graphs with bounded rankwidth, thus the strong chromatic index of Halin graphs can be computed in polynomial time. In passing, let us mention the following result. A class of graphs $\mathcal{G}$ is χ-bounded if there exists a function f such that $\chi(G) \leqslant f(\omega(G))$ for $G \in \mathcal{G}$. Here $\chi(G)$ is the chromatic number of G and $\omega(G)$ is the clique number of G. Recently, Dvořák and Král showed that for every k, the class of graphs with rankwidth at most k is χ-bounded [3]. Obviously, the graphs $L(G)^2$ have a uniform χ-bound for graphs G in the class of Halin graphs.

In this note we show that there exists a linear-time algorithm that computes the strong chromatic index of Halin graphs.

2 The strong chromatic index of Halin graphs

The following lemma is easy to check.

Lemma 1 (Ping-Ying Tsai). *Let* C_n *be the cycle with* n *vertices and let* W_n *be the wheel with* n *vertices in the cycle. Then*

$$s\chi'(C_n) = \begin{cases} 3 & \text{if } n = 0 \bmod 3 \\ 5 & \text{if } n = 5 \\ 4 & \text{otherwise} \end{cases} \qquad s\chi'(W_n) = \begin{cases} n+3 & \text{if } n = 0 \bmod 3 \\ n+5 & \text{if } n = 5 \\ n+4 & \text{otherwise.} \end{cases}$$

A double wheel is a Halin graph in which the tree T has exactly two vertices that are not leaves.

[5] The results of Ko-Wei Lih and Ping-Ying Tsai, *et al.*, were presented at the Sixth Cross-Strait Conference on Graph Theory and Combinatorics which was held at the National Chiao Tung University in Taiwan in 2011.

Lemma 2 (Ping-Ying Tsai). *Let W be a double wheel where x and y are the vertices of T that are not leaves. Then $s\chi'(T) = d(x) + d(y) - 1$ where $d(x)$ and $d(y)$ are the degrees of x and y. Furthermore,*

$$s\chi'(W) = \begin{cases} s\chi'(T) + 4 = 9 & \textit{if } d(x) = d(y) = 3, \text{ i.e., } \textit{if } W = \bar{C}_6 \\ s\chi'(T) + 2 = d(y) + 4 & \textit{if } d(y) > d(x) = 3 \\ s\chi'(T) + 1 = d(x) + d(y) & \textit{if } d(y) \geqslant d(x) > 3. \end{cases}$$

Let G be a Halin graph with tree T and cycle C. Then obviously,

$$s\chi'(G) \leqslant s\chi'(T) + s\chi'(C). \tag{1}$$

The linegraph of a tree is a claw-free blockgraph. Since a sun S_r with $r > 3$ has a claw, $L(T)$ has no induced sun S_r with $r > 3$. It follows that $L(T)^2$ is a chordal graph [9] (see also [2]; in this paper Cameron proves that $L(G)^2$ is chordal for any chordal graph G). Notice that

$$s\chi'(T) = \chi(L(T)^2) = \omega(L(T)^2) \leqslant 2\Delta(G) - 1 \quad \Rightarrow \quad s\chi'(G) \leqslant 2\Delta(G) + 4. \tag{2}$$

2.1 Cubic Halin graphs

In this subsection we outline a simple linear-time algorithm for the cubic Halin graphs.

Theorem 1. *There exists a linear-time algorithm that computes the strong chromatic index of cubic Halin graphs.*

Proof. Let G be a cubic Halin graph with plane tree T and cycle C. Let k be a natural number. We describe a linear-time algorithm that checks if G has a strong edge coloring with at most k colors. By Equation (2) we may assume that k is at most 10. Thus the correctness of this algorithm proves the theorem.

Root the tree T at an arbitrary leaf r of T. Consider a vertex x in T. There is a unique path P in T from r to x in T. Define the subtree T_x at x as the maximal connected subtree of T that does not contain an edge of P. If $x = r$ then $T_x = T$.

Let $H(x)$ be the subgraph of G induced by the vertices of T_x. Notice that, if $x \neq r$ then the edges of $H(x)$ that are not in T form a path $Q(x)$ of edges in C.

For $x \neq r$ define the boundary $B(x)$ of $H(x)$ as the following set of edges.

(a) The unique edge of P that is incident with x.
(b) The two edges of C that connect the path $Q(x)$ of C with the rest of C.
(c) Consider the endpoints of the edges mentioned in (a) and (b) that are in T_x. Add the remaining two edges that are incident with each of these endpoints to $B(x)$.

Thus the boundary $B(x)$ consists of at most 9 edges. The following claim is easy to check. It proves the correctness of the algorithm described below. Let e be an edge of $H(x)$. Let f be an edge of G that is not an edge of $H(x)$. If e and f are at distance at most 1 in G then e or f is in $B(x)$.[6]

Consider all possible colorings of the edges in $B(x)$. Since $B(x)$ contains at most 9 edges and since there are at most k different colors for each edge, there are at most

$$k^9 \leqslant 10^9$$

different colorings of the edges in $B(x)$.

The algorithm now fills a table which gives a boolean value for each coloring of the boundary $B(x)$. This boolean value is `TRUE` if and only if the coloring of the edges in $B(x)$ extends to an edge coloring of the union of the sets of edges in $B(x)$ and in $H(x)$ with at most k colors, such that any pair of edges in this set that are at distance at most one in G, have different colors. These boolean values are computed as follows. We prove the correctness by induction on the size of the subtree at x.

First consider the case where the subtree at x consists of the single vertex x. Then $x \neq r$ and x is a leaf of T. In this case $B(x)$ consists of three edges, namely the three edges that are incident with x. These are two edges of C and one edge of T. If the colors of these three edges in B are different then the boolean value is set to `TRUE`. Otherwise it is set to `FALSE`. Obviously, this is a correct assignment.

Next consider the case where x is an internal vertex of T. Then x has two children in the subtree at x. Let y and z be the two children and consider the two subtrees rooted at y and z.

The algorithm that computes the tables for each vertex x processes the subtrees in order of increasing number of vertices. (Thus the roots of the subtrees are visited in postorder). We now assume that the tables at y and z are computed correctly and show how the table for x is computed correctly and in constant time. That is, we prove that the algorithm described below computes the table at x such that it contains a coloring of $B(x)$ with a value `TRUE` if and only if there exists an extension of this coloring to the edges of $H(x)$ and $B(x)$ such that any two different edges e and f at distance at most one in G, each one in $H(x)$ or in $B(x)$, have different colors.

Consider a coloring of the edges in the boundary $B(x)$. The boolean value in the table of x for this coloring is computed as follows. Notice that

(i) $B(y) \cap B(z)$ consists of one edge and this edge is not in $B(x)$, and
(ii) $B(x) \cap B(y)$ consists of at most four edges, namely the edge (x, y) and the three edges of $B(y)$ that are incident with one vertex of $C \cap H(y)$. Likewise, $B(x) \cap B(z)$ consists of at most four edges.

[6] Two edges in G are at distance at most one if the subgraph induced by their endpoints is either P_3, or K_3 or P_4. We assume that it can be checked in constant time if two edges e and f are at distance at most one. This can be achieved by a suitable data structure.

The algorithm varies the possible colorings of the edge in $B(y) \cap B(z)$. Colorings of $B(x)$, $B(y)$ and $B(z)$ are consistent if the intersections are the same color and the pairs of edges in

$$B(x) \cup B(y) \cup B(z)$$

that are at distance at most one in G have different colors. A coloring of $B(x)$ is assigned the value `TRUE` if there exist colorings of $B(y)$ and $B(z)$ such that the three colorings are consistent and $B(y)$ and $B(z)$ are assigned the value `TRUE` in the tables at y and at z respectively. Notice that the table at x is built in constant time.

Consider a coloring of $B(x)$ that is assigned the value `TRUE`. Consider colorings of the edges of $B(y)$ and $B(z)$ that are consistent with $B(x)$ and that are assigned the value `TRUE` in the tables at y and z. By induction, there exist extensions of the colorings of $B(y)$ and $B(z)$ to the edges of $H(y)$ and $H(z)$. The union of these extensions provides a k-coloring of the edges in $H(x)$.

Consider two edges e and f in $B(x) \cup B(y) \cup B(z)$. If their distance is at most one then they have different colors since the coloring of $B(x) \cup B(y) \cup B(z)$ is consistent. Let e and f be a pair of edges in $H(x)$. If they are both in $H(y)$ or both in $H(z)$ then they have different colors. Assume that e is in $H(y)$ and assume that f is not in $H(y)$. If e and f are at distance at most one, then e or f is in $B(y)$. If they are both in $B(y)$, then they have different colors, due to the consistency. Otherwise, by the induction hypothesis, they have different colors. This proves the claim on the correctness.

Finally, consider the table for the vertex x which is the unique neighbor of r in T. By the induction hypothesis, and the fact that every edge in G is either in $B(x)$ or in $H(x)$, G has a strong edge coloring with at most k colors if and only if the table at x contains a coloring of $B(x)$ with three different colors for which the boolean is set to `TRUE`.

This proves the theorem. □

Remark 1. The involved constants in this algorithm are improved considerably by the recent results of Ko-Wei Lih, Ping-Ying Tsai, *et al.*.

2.2 Halin graphs of general degree

Theorem 2. *There exists a linear-time algorithm that computes the strong chromatic index of Halin graphs.*

Proof. The algorithm is similar to the algorithm for the cubic case.

Let G be a Halin graph, let T be the underlying plane tree, and let C be the cycle that connects the leaves of T. Since $L(T)^2$ is chordal the chromatic number of $L(T)^2$ is equal to the clique number of $L(T)^2$, which is

$$s\chi'(T) = \max \{ \ d(u) + d(v) - 1 \mid (u, v) \in E(T) \ \},$$

where $d(u)$ is the degree of u in the tree T. By Formula (1) and Lemma 1 the strong chromatic index of G is one of the six possible values[7]

$$s\chi'(T), s\chi'(T)+1, \ldots, s\chi'(T)+5.$$

Root the tree at some leaf r and consider a subtree T_x at a node x of T. Let $H(x)$ be the subgraph of G induced by the vertices of T_x. Let y and z be the two boundary vertices of $H(x)$ in C.

We distinguish the following six types of edges corresponding to $H(x)$.

1. The set of edges in T_x that are adjacent to x.
2. The edge that connects x to its parent in T.
3. The edge that connects y to its neighbor in C that is not in T_x.
4. The set of edges in $H(x)$ that have endpoint y.
5. The edge that connects z to its neighbor in C that is not in T_x.
6. The set of edges in $H(x)$ that have endpoint z.

Notice that the set of edges of every type has bounded cardinality, except the first type.

Consider a 0/1-matrix M with rows indexed by the six types of edges and columns indexed by the colors. A matrix entry M_{ij} is 1 if there is an edge of the row-type i that is colored with the color j and otherwise this entry is 0. Since M has only 6 rows, the rank over $GF[2]$ of M is at most 6.

Two colorings are equivalent if there is a permutation of the colors that maps one coloring to the other one. Let $S \subseteq \{1, \ldots, 6\}$ and let $W(S)$ be the set of colors that are used by edges of type i for all $i \in S$. A class of equivalent colorings is fixed by the set of cardinalities

$$\{\, |W(S)| \mid S \subseteq \{1, \ldots, 6\} \,\}.$$

We claim that the number of equivalence classes is constant. The number of ones in the row of the first type is the degree of x in $H(x)$. Every other row has at most 3 ones. This proves the claim.

Consider the union of two subtrees, say at x and x'. The algorithm considers all equivalence classes of colorings of the union, and checks, by table look-up, whether it decomposes into valid colorings of $H(x)$ and $H(x')$. An easy way to do this is as follows. First double the number of types, by distinguishing the edges of $H(x)$ and $H(x')$. Then enumerate all equivalence classes of colorings. Each equivalence class is fixed by a sequence of 2^{12} numbers, as above. By table look-up, check if an equivalence class restricts to a valid coloring for each of $H(x)$ and $H(x')$. Since this takes constant time, the algorithm runs in linear time.

This proves the theorem. $\square$

[7] Actually, according to the recent results of Ping-Ying Tsai, *et al.*, the strong chromatic index of G is at most $s\chi'(T)+3$ except when G is a wheel or $\bar{C}_6$.

References

1. Andersen, L., The strong chromatic index of a cubic graph is at most 10, *Discrete Mathematics* **108** (1992), pp. 231–252.
2. Cameron, K., Induced matchings, *Discrete Applied Mathematics* **24** (1989), pp. 97–102.
3. Dvořák, Z. and D. Král, Classes of graphs with small rank decompositions are χ-bounded. Manuscript on ArXiv: 1107.2161.v1, 2011.
4. Faudree, R., A. Gyárfás, R. Schelp, Z. Tuza, The strong chromatic index of graphs, *Ars Combinatorica* **29B** (1991), pp. 205–211.
5. Ganian, R. and P. Hliněný, Better polynomial algorithms on graphs of bounded rankwidth, *Proceedings IWOCA 09*, LNCS 5874 (2009), pp. 266–277.
6. Golumbic, M. and M. Lewenstein, New results on induced matchings, *Discrete Applied Mathematics* **101** (2000), pp. 157–165.
7. Halin, R., Studies on minimally n-connected graphs, in (D. Welsh ed.) *Combinatorial mathematics and its applications*, Academic Press (1971), pp. 129–136.
8. Ka, T., *The strong chromatic index of cubic Halin graphs*. M.Phil. Thesis, Hong Kong Baptist University, 2003.
9. Laskar, R. and D. Shier, On powers and centers of chordal graphs, *Discrete Applied Mathematics* **6** (1983), pp. 139–147.
10. Mahdian, M., *The strong chromatic index of graphs*. M.Sc. Thesis, Department of Computer Science, University of Toronto, 2000.
11. Mahdian, M., On the computational complexity of strong edge coloring, *Discrete Applied Mathematics* **118** (2002), pp. 239–248.
12. Molloy, M. and B. Reed, A bound on the strong chromatic index of a graph, *Journal of Combinatorial Theory, Series B* **69** (1997), pp. 103–109.
13. Salavatipour, M., A polynomial algorithm for strong edge coloring of partial k-trees, *Discrete Applied Mathematics* **143** (2004), pp. 285–291.
14. Shiu, W., P. Lam and W. Tam, On strong chromatic index of Halin graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **57** (2006), pp. 211-222.
15. Shiu, W. and W. Tam, The strong chromatic index of complete cubic Halin graphs, *Applied Mathematics Letters* **22** (2009), pp. 754–758.
16. Wu, J. and W. Lin, The strong chromatic index of a class of graphs, *Discrete Mathematics* **308** (2008), pp. 6254–6261.